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Diagonalisation of Idempotent Matrices

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0. INTRODUCTION

Let R be a ring, associative with unit element and with an involution $*$ on it. An $m \times n$ matrix A is said to have a Moore–Penrose (MP) inverse with respect to the involution $*$ iff there exists an $n \times m$ matrix X such that $AXA = A$; $XAX = X$; $(AX)^* = AX$; $(XA)^* = XA$. The solution, if it exists, is unique and denoted by A^+.

Several authors considered the problem of characterising matrices over certain domains for which an MP-inverse exists; cf. [1, 3, 6]. There results were generalised by Puystjens and Robinson; cf. [4]. The latter noted that if an $m \times n$ matrix A over a ring is of the form

$$A = (P_1\ P_2)\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

with I_r the $r \times r$ identity matrix, $P = (P_1, P_2)$ and $(Q_1\ Q_2)^{\mathrm{T}}$ invertible matrices, then A has an MP-inverse iff $P_1^* P_1$ and $Q_1 Q_1^*$ are invertible.

The MP-inverse is then given by the formula

$$A^+ = Q_1^*(Q_1 Q_1^*)^{-1}(P_1^* P_1)^{-1} P_1^*.$$

(In the sequel it is denoted by the Puystjens–Robinson characterisation). It is not clear yet what is the largest class of rings for which the Puystjens–Robinson characterisation holds. A necessary condition is that certain von Neumann regular matrices A, i.e., there is a matrix X such that $AXA = A$, are diagonisable and therefore especially certain idempotent matrices must be diagonisable. This motivated us to consider rings over which all idempotent matrices are diagonisable, i.e., ID-rings. We discuss these rings in Section 1. In Section 2 we consider the relation between diagonalisation of idempotent matrices and diagonalisation of von Neumann regular matrices.

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The class of ID-rings is still too restrictive. In [6] Rao considered rings which are not ID but for which the Puystjens–Robinson characterisation still holds. In Section 3 we discuss a large class of rings containing ID-rings and rings satisfying Rao's condition, namely ID-*rings, i.e., rings over which *-symmetric idempotent matrices are diagonisable. We prove that this class of rings is much larger, and even enables us to handle, Moore–Penrose inverse problems over certain Dedekind domains. ID-*rings may turn out to be the widest class for which the above characterisation of matrices having an MP-inverse holds.

We are most grateful to R. Puystjens, for the many discussions we had together on the subject.

1. ID-Rings

R will denote a ring, not necessarily commutative, with unit element 1.

Definition 1.1. A ring R is called an *ID-ring* if every idempotent matrix over R is diagonisable; i.e., for all $E \in M_n(R)$, $n \in \mathcal{N}$, $E^2 = E$ there exists invertible matrices P, $Q \in M_n(R)$ such that $PEQ = D = \operatorname{diag}(e_1, ..., e_r, 0, ..., 0)$, $D^2 = D = D'$.

We recall some of the main facts of ID-rings

(1) If R is a domain then R is an ID-ring iff R is projective free (i.e., every finitely generated projective module is free).

This is obvious, considering the image of an idempotent operator on R^n which is a projective module and its direct complement in R^n is the kernel of the operator. Most of the known ID-rings are rings which are projective free. One must keep in mind that it is a rather difficult problem to decide whether a given domain is projective free or not, cf. [3].

Proposition 1.4 gives some information on ID-rings in general.

(2) For commutative rings R, Steger obtained the following results, cf. [9].

- (a) In a commutative ID-ring every stable free module is free (i.e., if M is a projective module such that $M \oplus R^n = R^m$ then M is said to be stable free).
- (b) In a commutative ID-ring every invertible ideal is principal. So, as a consequence, any Dedekind domain which is not a principal ideal domain cannot be an ID-ring.
- (c) The following classes of commutative rings are ID-rings:

(i) T-regular rings,
(ii) quasi-semi-local rings,
(iii) Artinian rings,
(iv) Polynomial rings in one variable over principal ideal rings.

We give a local characterisation of commutative ID-rings. It is based on Vaserstein's proof of the Serre conjecture.

Let R be a commutative ring, S a multiplicatively closed set in R not containing 0. With R_S we denote the localisation of R with respect to S, i.e., $R_S = \{(a, s) \mid a \in R, s \in S\}$/equivalence relation $\sim_S$, where $(a, s) \sim_S (a', s')$ iff $s''(as' - a's) = 0$ for some $s'' \in S$.

LEMMA 1.2 (*Vaserstein*). *Let R be a commutative ring, S a multiplicatively closed set in R. Let $P(x) \in GL_n(R_S[x])$ such that $P(0) = I_n$. Then there exists a matrix $\hat{P}(x)$ in $GL_n(R[x])$ such that $\hat{P}(x)$ localises onto $P(sx)$ for some $s \in S$, and $\hat{P}(0) = I_n$.*

Proof. cf. [3].

PROPOSITION 1.3. *Let R be a commutative ring and S a multiplicatively closed set in R. Let $E(t)$ be an idempotent matrix over $R[t]$; the following statements are equivalent:*

(1) *$E(t) \approx E(0)$ over $R_S[t]$ (i.e., $\exists P(t)$ invertible matrix over $R_S[t]$ with $P(t)\, E(t)\, P^{-1}(t) = E(0)$).*

(2) *There exists a $b \in S$ such that $E(t + bx) \approx E(t)$ over $R[t, x]$.*

Proof. (2) $\Rightarrow$ (1). Consider the congruence $E(t + bs) \approx E(t)$ over $R_S[t, x]$ and specialise to $t \mapsto 0$, $x \mapsto b^{-1}t$. One gets $E(0 + bb^{-1}t) = E(t) \approx E(0)$ over $R_S[t]$.

(1) $\Rightarrow$ (2). Take $Q(t) \in GL_n(R_S[t])$ such that

$$Q(t)^{-1} E(t)\, Q(t) = E(0)$$

Let $P(t, x) = Q(t + x)\, Q(t)^{-1} \in GL_n(R_S[t, x])$. Then

$$\begin{aligned} &P(t, x)^{-1} E(t + x)\, P(t, x) \\ &\quad = Q(t) \cdot Q(t + x)^{-1} E(t + x)\, Q(t + x)\, Q(t)^{-1} \\ &\quad = Q(t)\, E(0)\, Q(t)^{-1} = E(t). \end{aligned}$$

This equation holds over $R_S[t, x]$.

Since $P(t, 0) = Q(t)\, Q(t)^{-1} = I_n$. We apply the lemma over $R[t]$. So there is a $\hat{P}(t, x)$ in $GL_n(R[t, x])$ localising to $P(t, sx)$ for some $s \in S$ and $\hat{P}(t, 0) = I_n$. Then over $R[t, x]$ one has

$$\hat{P}(t, x)^{-1} E(t + dx)\, \hat{P}(t, x) - E(t) = xF(t, x)$$

with $F(t, x)$ a matrix localising to 0. So for suitable $s' \in S$,

$$\hat{P}(t, x'x)^{-1} E(t + ss'x)\, \hat{P}(t, s'x) - E(t) = 0.$$

PROPOSITION 1.4. *Let R be a commutative ring and $E(t)$ an idempotent matrix over $R[t]$. If $E(t) \approx E(0)$ over $R_{\mathscr{M}}[t]$ for all maximal ideals $\mathscr{M} \subset R$ then $E(t) \approx E(0)$ over $R[t]$.*

Proof. Let $I = \{a \in R \mid E(t) \approx E(0)$ over $R_a[t]\}$ (R_a is the localisation of R at $\{1, a, a^2, \dots\}$) and $J = \{b \in R \mid E(t + bx) \approx E(t)$ over $R[t, x]\}$. Then I, J are ideals in R and $I = \operatorname{rad} J$. This because if $b \in J$, $c \in R$ then substituting x by cx gives $E(t + bcx) \approx E(t)$ so $bc \in J$. Also $b, b' \in J$ then $b + b' \in J$ follows after substituting t by $t + b'x$. Proposition 1.3 yields that $I = \operatorname{rad} J$. Now for any maximal ideal $\mathscr{M} \subset R$, there is a $b \in R \setminus \mathscr{M}$ with $b \in J$. So $I = J = R$, therefore $E(t) \approx E(0)$ over $R[t]$.

COROLLARY 1.5. *If R is an* ID*-ring then $R[t]$ is an* ID*-ring iff $R_{\mathscr{M}}[t]$ is* ID *for all maximal ideals $\mathscr{M}$ in R.*

Remark. Steger indicates the existence of examples of local rings R such that $R[t]$ is not ID. However, in general, being ID should be easier to decide for $R_{\mathscr{M}}[t]$ then for $R[t]$.

2. VON NEUMANN REGULAR MATRICES OVER ID-RINGS

In this section we consider the following problem: Let R be a ring with unit and A a von Neumann regular matrix over R. Under which conditions is A diagonisable. We know that to a von Neumann regular matrix one can associate in a natural way (many) idempotent matrices. Namely if X is a von Neumann regular inverse then $A = AXA$ so AX is idempotent. Furthermore $\operatorname{Im} A \cong \operatorname{Im} AX$ and consequently, since both are direct summands of a free module,

$$\operatorname{coker} A \cong \operatorname{coker} AX.$$

In view of this one might expect that if AX is diagonisable to an idempotent matrix so is A. In general this is not true. In [5] there is a counterexample over the Weyl algebra. However, we are tempted to make the following conjecture.

If R is an ID-*ring then every von Neumann regular matrix over R is diagonisable*!

For domains the conjecture follows from the Introduction. Also a large class of rings R satisfy the following property (*):

If A, B are $n \times m$ matrices over R and coker $A \cong$ coker B then there exist invertible matrices P, Q such that $PAQ = B$. (A survey on rings satisfying (*) can be found in [10].) By the above remark the conjecture holds for all ID-rings satisfying (*). For domains and for Artinian rings (the latter satisfy (*)), we give direct proofs of this fact. These proofs have the advantage that an explicit diagonalisation method is given.

PROPOSITION 2.1. *Let R be an Artinian ring. Let $A \in M_n(R)$, which is von Neumann regular. Then there exist invertible matrices P, $Q \in M_n(R)$ such that $A = PDQ$, with $D^2 = D = D^t$, and D a diagonal matrix.*

Proof. By Wedderburn's structure theorem we have that $A = TD_0S + A_1$, with T, S invertible, $D_0^2 = D_0 = D_0^t =$ diagonal and A_1 a matrix over rad R (the Jacobson radical of R). Let $A' = T^{-1}AS^{-1}$ then

$$A' = \begin{pmatrix} D_0 + J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \quad \text{with} \quad D_0 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

J_i matrices over rad R. So

$$A' = \begin{pmatrix} D(1 + DJ_1) + (1 - D)\,J_1 & J_2 \\ J_3 & J_4 \end{pmatrix},$$

with $1 + DJ_1$ invertible. Take

$$T_1 = \begin{pmatrix} (1 + DJ_1)^{-1} & -(1 + DJ_1)^{-1}J \\ 0 & 1 \end{pmatrix}$$

then

$$A'T_1 = \begin{pmatrix} D + (1 - D)\,J_1(1 + DJ_1)^{-1} & -DJ_2 - (1 - D)\,J_1(1 + DJ_2)\,J_2^{-1} + J_2 \\ J_3(1 + DJ_1)^{-1} & -J_3(1 + DJ_1)^{-1}J_2 + J_4 \end{pmatrix}$$

$$= \begin{pmatrix} (1 + (1 - D)\,J_1(1 + DJ_1)^{-1})\,D + (1 - D)\,J_1(1 + DJ_1)^{-1}(1 - D) & \\ & (1 - D)\,J_2(1 - D)\,J_1(1 + DJ_1)\ {}^{1}J_2 \\ J_3(1 + D)\,J_1)^{1} & J_4 - J_3(1 + DJ_1)^{-1}J_2 \end{pmatrix}.$$

Also $(1 + (1 - D)\,J_1(1 + DJ_1)^{-1})$ is invertible.

Multiplying on the left with

$$T_2 = \begin{pmatrix} [1+(1-D)\,J_1(1+DJ_1)^{-1}]^{-1} & 0 \\ -J_3(1+DJ_1)^{-1}[1+(1-D)\,J_1(1+DJ_1)^{-1}]^{-1} & 1 \end{pmatrix}$$

and noting that

$$\begin{aligned}[1+(1-D)\,J_1(1+DJ_1)^{-1}]^{-1} \\ &= 1-(1-D)\,J_1(1+DJ_1)^{-1}[1+(1-D)\,J_1(1+DJ_1)^{-1}]^{-1}\end{aligned}$$

we obtain

$$T_2A'T_1 = \begin{pmatrix} D+(1-D)\,J_1'(1-D) & (1-D)\,J_2' \\ J_3'(1-D) & J_4' \end{pmatrix}$$

with J_1', J_2', J_3', J_4' matrices over rad R. Or $T_2A'T_1 = S+(1-S)\,J'(1-S)$ with

$$S = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad J' = \begin{pmatrix} J_1' & J_2' \\ J_3' & J_4' \end{pmatrix}.$$

Since A is von Neumann regular so is $T_2A'T_1$, therefore there exists a matrix X with

$$(S+(1-S)\,J'(1-S)\,X(S+(1-S)\,J'(1-S)) = S+(1-S)\,J'(1-S)$$

But $X = (S+(1-S))\,X(S+(1-S))$ so

$$\begin{aligned}SXS + SX(1-S)\,J'(1-S) + (1-S)\,J'(1-S)\,XS \\ + (1-S)\,J'(1-S)\,X(1-S)\,J'(1-S) = S+(1-S)\,J'(1-S)\end{aligned}$$

Multiplying on the left- and right-hand side with $(1-S)$ yields:

$$(1-S)\,J'(1-S)\cdot X(1-S)\,J'(1-S) = (1-S)\,J'(1-S)$$

i.e., $(1-S)\,J'(1-S)$ is a von Neumann regular matrix in rad R therefore $(1-S)\,J'(1-S)=0$. Finally $T_2A'T_1 = T_2T^{-1}AS^{-1}T_1 = \left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Remark. Modulo the structure theorem the proof gives a procedure to diagonalise a von Neumann regular matrix. The reduction method from A to $TDS+J$ use the classical algorithm for matrices over principal ideal rings cf. [2]. Note that although the calculations are rather lengthy the proof is completely elementary.

PROPOSITION 2.2. *Let R be an* ID-*domain. Any von Neumann regular* *matrix* $A \in M_n(R)$ *is diagonalisable to* diag(1,..., 1, 0,..., 0), *i.e. there exist invertible matrices* P, Q *such that*

$$A = P \begin{pmatrix} 1 & & \\ & 1 & 0 \\ 0 & & 0 \end{pmatrix} Q.$$

Proof. First, note that if E is an idempotent matrix then $E = S\mathrm{Diag}(1, 1,..., 1, 0...0)\, S^{-1}$, with S invertible. This follows since Im E and Ker E are projective and therefore free. Choosing suitable bases in Im E + Ker E yields S as transformation matrix.

Let $X \in M_n(R)$ such that $A \times A = A$ then $AX = TD_1T^{-1}$ and $XA = S^{-1}D_2S^{-1}$, with T, S invertible matrices and D_1 and D_2 diagonal matrices. Therefore,

$$\begin{aligned} T^{-1}AS^{-1} &= T^{-1}AXAS^{-1}SXAS^{-1} \\ &= T^{-1}AS^{-1}D, \end{aligned}$$

analogously $T^{-1}AS^{-1} = D_1T^{-1}AS^{-1}$. Now divide T^{-1}, S^{-1} into block matrices of dimension equals the number of 1's in D_1, respectively, D_2, i.e., equals rg. Im AX (resp. rg Im XA), both $=$ rg A. We obtain

$$\begin{aligned} T^{-1}AS^{-1} = T^{-1}AS^{-1}D_2 = \begin{pmatrix} T'_1 \\ T'_2 \end{pmatrix} A \quad (S'_1 \quad S'_2) \quad D_2 \\ \times \begin{pmatrix} T'_1AS'_1 & 0 \\ T'_2AS'_1 & 0 \end{pmatrix}. \end{aligned}$$

Also

$$T^{-1}AS^{-1} = \begin{pmatrix} T'_1AS'_1 & T'_1AS'_2 \\ 0 & 0 \end{pmatrix}.$$

(Hence

$$T^{-1} = \begin{pmatrix} T'_1 \\ T'_2 \end{pmatrix}, S^{-1} = (S'_1 \ S'_2)).$$

Comparing both equations yields $T'_1AS'_2 = 0 = T'_2AS'_1$. So

$$A = T\begin{pmatrix} T'_1AS_1 & 0 \\ 0 & 0 \end{pmatrix} S.$$

Note that $T'_1 AS'_1$ is an $r \times r$ matrix with $r = \operatorname{rg} \operatorname{Im} A$. Since A is von Neumann regular so is $T'_1 AS'_1$, on the other hand it is an $r \times r$ matrix with its image of rang r so since R is a domain it has to be invertible. Therefore

$$A = T\begin{pmatrix} (T'_1 AS'_1) & 0 \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} S$$

$$= P\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix} Q, \qquad \text{with } P, Q \text{ invertible.}$$

3. ID*-Rings

In [6] Rao proofs that the Puystjens-Robinson characterisation for matrices, over an integral domain, having an MP-inverse with respect to transposition t, also holds for a class of domains which are not necessarily ID. He introduces the following condition on a ring R: If $a_1, ..., a_n \in R$ then $a_1 = a_1^2 + \cdots + a_n^2$ implies $a_2 = a_3 = \cdots = a_n = 0$. As an application he considers polynomial rings $Z[X_1, ..., X_n]$.

First, we note that the condition is also satisfied for every integral closure of the integers in a number field which is imbeddable in the real numbers $\mathbb{R}$. It then follows also for polynomial rings, $R[X_1, ..., X_n]$, over such integral closures R, e.g., $R = Z[\sqrt{d}]$ where $d \geqslant 1$ or $R = Z[\sqrt{d}][X_1 \cdots X_n]$.

Rao's condition looks rather artificial but it is easy to explain:

If a matrix A has an MP-inverse A^+, then AA^+ is a idempotent symmetric with respect to the involution considered. So in the above cases, since one considers the transposition as the involution on R, AA^+ is symmetric for transposition:

If $AA^+ = (a_{ij})$ then it follows that

$$\sum_j a_{ij}^2 = a_{i1} \qquad \text{for all } i.$$

So there are now symmetric idempotents except for the diagonal matrices with only 1's or 0's on the diagonal if Rao's condition is satisfied, so AA^+ is permutation equivalent to diag(1,..., 1, 0...0). Let R be a ring with involution * then

Definition 3.1. The ring R is an ID-* ring if every *-symmetric idempotent is diagonalisable; i.e., If $E \in M_n(\mathscr{R})$ with $E^* = E$ and $E^2 = E$ then there exist invertible matrices P, $Q \in M_n(R)$ such that $PEQ = \operatorname{diag}(e_1, ..., e_r, 0...0)$

COROLLARY 3.2. *A commutative ring satisfying Rao's condition; $a_1^2 + \cdots + a_n^2 = a$, implies $a_2 = \cdots = a_n = 0$; is an* ID-t *ring, where t denotes transposition of matrices.*

Proof. Trivial from the above remark.

We will show that ID-* rings form a much wider class than ID-rings and rings satisfying Rao condition. Let us recall some facts about Dedekind domains. A ring R which is Noetherian integrally closed and every ideal $I \neq D$ is the product of prime ideals in a unique way is called a Dedekind domain. Let R be a Dedekind domain, K its field of quotients. An R-submodule of K, M for which there is an element $r \neq 0 \in R$ such that $rM \subset R$, is called a fractional ideal of R. In a Dedekind domain the set of these fractional ideals form agroup $\mathscr{F}(R)$, it is the free abelian group generated by the prime ideals of R. Let $I \in F(R)$ then I^{-1} is given by the set $\{x \in K \mid xI \subset R\}$, so $II^{-1} = R$. Note that if $\gamma: I \to R$ is an R-homomorphism then it extends to an R-homomorphism $\tilde{\gamma}: K \to K$, so it is given by multiplication with an element $x \in K$, clearly $x \in I^{-1}$. So the map: $I^{-1} \to \operatorname{Hom}(I, R)$. $x \mapsto$ multiplication with x defines an isomorphism (injectivity follows from the fact that R is a domain, therefore multiplication acts faithfully), between $I^{-1}$and the R-dual of I, $I^{-1} = \operatorname{Hom}(I, R)$.

If one considers isomorphism classes of fractional ideals one gets the ideal class group $C(R)$ which is isomorphic to $\mathscr{F}(k)$/principal ideals by an analogous argument as above. Finally every ideal of a Dedekind domain is projective. This follows from the fact that it is invertible, $II^{-1} = R$ gives rise to a dual base for I.

Let $K_0(R)$ be the Grothendieck group of R, i.e., take G the free abelian group generated by the isomorphism classes, (P), of finitely generated projective modules P over R, and H the subgroup of G generated by $(P \oplus Q) - (P) - (Q)$. Then $K_0(R) = G/H$.

For a Dedekind domain R one has:

(1) Every finitely generated projective module $P = R \oplus \cdots \oplus R \oplus I$ with I an ideal in R.

(2) $K_0(R) \cong Z \oplus \operatorname{Cl}(R)$.

(For proofs and further details we refer to [7].) We now prove

THEOREM 3.3. *Let R be a Dedekind domain and* $\operatorname{Cl}(R)$ *its class group. If* $\operatorname{Cl}(R)$ *contains no element of order* 2 *then R is an* ID-t *ring.*

Proof. Let A be an idempotent $n \times n$ matrix over R symmetric with respect to t. Choose $\{e_1,..., e_n\}$ a basis for R^n. Let α be the linear homomorphism defined by A with respect to $\{e_1,..., e_n\}$. Take $\{e_1^*,..., e_n^*\}$ a dual basis of $\{e_1,..., e_n\}$ in $\mathrm{Hom}(R^n, R)$. One has

$$\begin{tikzcd} R^n \arrow[r, "A"] \arrow[d, "\wr" description, no head] & R^n \arrow[d, "\wr" description, no head] \\ \mathrm{Hom}(R^n, R) \arrow[r, "A^t"] & \mathrm{Hom}(R^n, R) \end{tikzcd}$$

Since $A = A^t$ it follows that $\mathrm{Im}\, A \cong \mathrm{Im}\, A^t$, because for $\forall i$ if $A(e_i) = \sum \alpha_i e_i$ then $A^t(e_i^*) = \sum \alpha_i e_i^*$. But $\mathrm{Im}\, A^t \cong \mathrm{Hom}(\mathrm{Im}\, A, R)$, which follows from

$$\mathrm{Hom}(\mathrm{Im}\, A \oplus \mathrm{Ker}\, A, R) \cong \mathrm{Hom}(\mathrm{Im}\, A, R) \oplus \mathrm{Hom}(\mathrm{Ker}\, A, R)$$

and the fact that elements of $\mathrm{Im}\, A^t$ act trivally on $\mathrm{Ker}\, A$. Now $\mathrm{Im}\, A$ is a finitely generated projective module so $\mathrm{Im}\ A = R + \cdots + R + I$ and $\mathrm{Hom}(\mathrm{Im}\, A, R) \cong R + \cdots + R + \mathrm{Hom}(I, R)$ with I an ideal in R. So in $K_0(R)$ since $-[I] = [I^{-1}] = [\mathrm{Hom}(I, R)]$ we obtain $[R^r + I] = [R^r + I^{-1}]$, or $[I] = -[I]$, i.e., $[I]$ is 2-torsion in $K_0(R)$ and therefore in $\mathrm{Cl}(R)$. By the hypothesis it follows that I is a principal ideal. But then $\mathrm{Im}\, A$ is a free module, choosing suitable basis then yields that A is diagonalisable.

Remark. (1) In general the condition in Theorem 3.3 is not very strong. Since there exists for every abelian group G a Dedekind domain with G as class group, theorem 3.3 yields that there are plenty of ID-t rings. However, for rings of integers in number fields very "often" one has that 2 divides the order of the class group. But examples in nonreal number fields do exist, so the class of rings is still more extensive than the one satisfying Rao's condition.

(2) The proof holds more generally for rings for which $K_0(R)$ has the above structure, e.g., Krull domains, etc.

REFERENCES

1. D. BATIGNE, Integral generalized inverses of integral matrices, *Linear Algebra Appl.* **22** (1978), 125–134.
2. P. COHN, "Free Rings and Their Relations," Academic Press, London, 1971.
3. T. Y. LAM, "Serre's Conjecture," Lecture Notes in Math., No. 635, Springer, Berlin, 1978.
4. R. PUYSTJENS AND D. W. ROBINSON, The Moore–Penrose inverse of a morphism with factorisation, *Linear Algebra Appl.* **30** (1981), 129–141.

5. R. PUYSTJENS AND J. VAN GEEL, On the diagonalisation of von Neumann regular matrices, *Acta Univ. Carolinae Math. et Phys.* **26** (2) (1985), 51–56.
6. B. RAO, On generalized inverses of matrices over integral domains, *Linear Multilinear Algebra* **10** (1981), 145–154.
7. I. REINER, "Maximal Orders," Academic Press, New York, 1975.
8. E. SONTAG, On generalised inverses of polynomial and other matrices, *IEEE Trans. Automat. Control* **25** (3) (1980), 514–517.
9. A. STEGER, Diagonability of Idempotent Matrices. *Pacific J. Math.* **19** (3) (1966), 535–542.
10. R. WARFIELD, Stable equivalence of matrices and resolutions, *Comm. Algebra* **6** (17) (1978), 1811–1828.